Finite Element Analysis of Nonlinear Models in Option Pricing

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Abstract

In this project, we will explore current nonlinear stochastic nonlinear stochastic differential equations used in option pricing and financial business delay decisions. We will study the well-posedness of the discrete problems, provide stability and convergence analysis of some finite element schemes with numerical examples.

1. Introduction

The nonlinear PDE models for pricing derivative securities in the presence of both transaction costs as well as the risk are currently being studied by many. Most authors use finite difference methods to solve the problems numerically. Finite element method for spatial discretization and the Crank-Nicolson method for time integration or other methods such as the time-space finite elements have been used to solve some problems numerically. The models are first transformed by the change of variables \( H = S\delta^2 V \), \( x = \ln(S/E) \), and the backward time \( \tau = T - t \) into a nonlinear heat equation in \( H \) with initial and boundary conditions and then the time-spatial approximation schemes are imposed on the heat equation. For example the RAPM model and LeLand’s model are the two important models under consideration in this project. In [2], the transformed model was studied numerically by using a stable full space-time discretization scheme based on the finite-volume method [3]. It has been shown that by using just a few linear and quadratic finite elements for of the RAPM model, the numerical solutions are compared with those in literature favorably [6]. Therefore, it is desirable to have some numerical analysis of the underlying finite element method.

2. The RAPM model and the project goals

Risk-adjusted Black-Scholes equation:

\[
\partial_t V + \frac{\sigma^2}{2} S^2 [1 - \mu(S\delta^2 V)^{\frac{1}{3}}] \delta^2 V = r(V - S\delta V),
\]

where \( \mu = 3 \left( \frac{C^2 R}{2\pi} \right)^{\frac{1}{3}} \).

Note that (1) is the backward parabolic PDE if and only if the function:

\[
\beta(H) = \frac{\sigma^2}{2} \left( 1 - \mu H^\frac{1}{3} \right) H
\]

is an increasing function in the variable \( H \), where

\[
H := S\Gamma = S\delta^2 V
\]
By the substitution (2) and (3), we get

\[ \partial_t H + S \partial_x^2 (S \beta(H)) = r(H - S \partial_x^2 (S \partial_x V)) \]

Using the following change of variables

\[ x := \ln \left( \frac{S}{E} \right), \quad x \in \mathbb{R} \]
\[ \tau := T - t, \quad \tau \in (0, T) \]

and after some simplifications we obtain the H equation in the new spatial and time variables:

\[ \partial_\tau H = \partial_x^2 \beta(H) + \partial_x \beta(H) + r \partial_x H \] (4)

The quasilinear parabolic equation above is also called H equation. This equation is subjected to the following boundary conditions:

\[ H(-\infty, \tau) = H(\infty, \tau) = 0, \quad \tau \in (0, T) \]

So as to solve the H equation we will use the discretization technique. Let us define \( x \in (-L, L) \). Because of the change of the variable, we can write \( S \) as

\[ S = Ee^x \]

Taking into account the interval of \( x \), we see that

\[ S \in (Ee^{-L}, Ee^L). \]

Consequently, we have to adjust the boundary conditions for \( H \). Thus,

\[ H(-L, \tau) = H(L, \tau) = 0, \quad \tau \in (\tau_*, T) \] (5)

This project will focus on problems like the one defined by (4) and (5) for several nonlinear option pricing with stocks or bonds as underlying properties [1-2] and optimal strategies for business decisions, such as delayed decisions [4-6]. The first goal is to generate a generate a finite element code with arbitrary number of elements and the second goal is to study the existence, uniqueness, and stability of the discrete problems including convergence and error estimates.

References